Optimal Control of DC Motor using Linear Quadratic Regulator

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Abstract: This paper provides the implementation of optimal control for an armature-controlled DC motor. The selection of error weighted Matrix and control weighted matrix in order to implement optimal control theory for improving the dynamic behavior of DC motor is presented. The closed loop performance of Armature controlled DC motor with derived linear optimal controller is then evaluated for the transient operating condition (starting). The result obtained from MATLAB is compared with that of PID controller and simple closed loop response of the motor.

Keywords: Optimal Control, Dc motor, Performance Index, MATLAB.

Introduction

With exploration of space research and missile technology, the complexity and demand of high accuracy for space probe and missiles introduced the world optimal control theory during the 1970s. At that time the challenge to scientists and engineers was to minimise the weight of satellites and missiles and control them accurately. Ref. [1] The linear quadratic control problem has its origins in the celebrated work of N.Wiener on mean –square filtering for weapon fire control during World war II (1940-45) . Ref. [2] Wiener solved the problem of designing filters that minimize a mean square error criterion of the form

$$J = E\{e^2(T)\}$$

Where e(t) is the error, and E{x} represents the expected value of the random variable x . Ref. [3] For a deterministic case, the above error criterion is generalized as an integral quadratic term as

$$J = \int e'(t)\, Q\, e(t)\, dt$$

Where, Q is some positive definite matrix. R. Bellman in 1957 introduced the technique of dynamic programming to solve discrete-time optimal control problems. Ref. [4] But, the most important contribution to optimal control systems was made in 1956 by L S Pontryagin and his associates, in development of his celebrated maximum principle Ref. [5]. In United States, R.E Kalman in 1960 provided linear quadratic regulator (LQR) and Linear quadratic Gaussian (LQG) theory to design optimal feedback controls. Ref. [6][7] He went on to present optimal filtering and estimation theory leading to his famous discrete Kalman filter and the continuous Kalman filter with Bucy Ref. [8]. Kalman had a profound effect on optimal control theory and the Kalman filter is one of the most widely used technique in applications of control theory to real world problems in variety of fields.

This paper presents optimal control of a DC motor such that it improves the transient response of the DC motor and reduces the steady state error in the motor speed. Section 2 details the optimal control theory. Section 3gives the state space model of DC motor derived from machine equations and fundamental laws of science. Section 4 compares the closed loop performance of motor without compensation, with PID Control action and with linear quadratic regulator. Section 5 summarizes all the significant conclusions drawn from the paper.

Optimal Control Theory

The main objective is to determine control signals that will cause a process(plant) to satisfy some physical constraints and at the same time extremize (maximize or minimize) a chosen performance criterion (performance index or cost function).Ref. [8] A designer aims at finding the optimal control vector that will drive the plant P from initial state to final state with some constraints on controls and states and at the same time extremizing the given performance index J .

The formulation of optimal control problem requires

1. A mathematical description (or model) of the process to be controlled (generally in state variable form),
2. A specification of performance index, and

3. A statement of boundary conditions and the physical constraints on the states and /or controls.

Performance Index

The typical performance criteria in classical control design are system time response to step or ramp input characterized by rise time, settling time, peak overshoot, and steady state accuracy; and the frequency response of the system characterized by gain and phase margins, and bandwidth.

Performance Index (PI) can be calculated or measured and used to evaluate system performance. Ref. [8] System performance is needed to be observed because of the need for – parameter optimization, design of optimum system and operation of modern adaptive control system.

The optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable or trajectory and at the same time extremize a performance index(PI) which may take several forms described as :

1. PI for Time-optimal Control system: In minimum time system to be transferred from an arbitrary initial state $x(t_0)$ to a specified final state $x(t_f)$.

$$J = \int_{t0}^{t} dt$$

2. PI for fuel-optimal Control system: For minimization of the total expenditure of fuel. Assume magnitude $|u(t)|$ is proportional to the rate of fuel consumption, where $|u(t)|$ is the thrust of rocket engine.

$$J = \int_{t0}^{t} |u(t)|\ dt$$

3. Performance Index for Minimum-Energy Control system: For minimization of the total expended energy. Consider a resistive circuit where R_i is the resistance of the i_{th} loop and u_i is the current in that loop.

$$J = \int_{t0}^{t} \sum_{i=1}^{m} R_i\ |u_i(t)|\ dt$$

4. PI for terminal Control system: For minimization of error between the desired target position $x_d(t_f)$ and the actual target $x_a(t)$ at the final time t_f.
5. PI for general optimal control system

$$J = x'(t_f)Fx(t_f) + \int_{t_0}^{t_f} [x'(t)Qx(t) + u'(t)Ru(t)]\, dt$$

Where R is a positive definite matrix, Q and F are positive semi-definite matrix respectively. Q and R may be time varying. This particular form is called quadratic (in terms of states and controls)form.

Constraints

The control u(t) and state x(t) vectors are either unconstrained or constrained depending upon the physical situation. The unconstrained problem is less involved and gives rise to some results. From the physical considerations, we have controls and states, such as currents and voltages in an electrical circuit, speed of a motor, thrust of a rocket, constrained as

$$U_+ \leq u(t) \leq U_- \text{ and } X_- \leq x(t) \leq X_+$$

Where + and – indicate the maximum and minimum values the variables can attain.

Formal Statement Of Optimal Control System

The optimal control problem is to find the optimal control $u^*(t)$ (* indicates extremal or optimal value) which causes the linear time-invariant plant (system)

$$\dot{x}(t) = A\, x(t) + B\, u(t)$$

to give the trajectory $\dot{x}(t)$ that optimizes or extremizes(minimizes or maximizes) a performance index

$$J = x'(t_f)\, F\, x(t_f) + \int [\, x'(t)\, Q\, x(t) + u'(t)\, R\, u(t)\,]dt$$

Or which causes the non linear system $\dot{x}(t) = f(x(t),u(t),t)$ to give the state $x^*(t)$ that optimizes the general performance index

$$J = S(x(t_f),t_f) + \int V(x(t),u(t),t)dt$$

With some constraints on the control variable u(t) and/or the state variables x(t) . The final time t_f may be fixed or free, and the final (target) state may be fully or partially fixed or free.

Ref. [8] The optimal control systems are studied in three stages:

1. In the first stage, we consider the performance index and use the well known theory of calculus of variations to obtain optimal functions.
2. In second stage, we bring in the plant and try to address the problem of finding optimal control $u^*(t)$ which will drive the plant and at the same time optimize the performance index.
3. Finally, the topic of constraints on the controls and states is considered along with the plant and performance index to obtain optimal control.

Different Type of Systems

Different cases depending on the statement of problem regarding the final time t_f and the final state $x(t_f)$

1. Type (a) Fixed- final time and fixed- final state system: Here, since t_f and $x(t_f)$ are fixed or specified and there is no extra boundary condition to be used other than those given in the problem formulation.
2. Type (b) Free-final Time and fixed-final state system: t_f is free or not specified in advance, δt_f is arbitrary and $x(t_f)$ is fixed or specified.
3. Type (c) Fixed-final time and free-final state system: Here t_f is specified and $x(t_f)$ is free.
4. Type (d) Free-final time and dependent free-final state system : t_f and $x(t_f)$ are related such that $x(t_f)$ lies on a moving curve with respect to t_f.
5. Type (e) Free-final time and Independent Free-final state: t_f and $x(t_f)$ are not related.

Open-loop Optimal Control

One has to construct or realize an open-loop optimal controller and in most cases it is very tedious. Also, changes in plant parameters are not taken into account by open-loop optimal controller which prompts us to think in terms of closed loop optimal controller i.e. to obtain optimal $u^*(t)$ in terms of state $x^*(t)$. Ref. [8] This closed loop optimal controller will have advantages such as sensitive to plant parameter variations and simplified construction of the controller.

Matrices in Performance Indices

1. The error weighted matrix $Q(t)$: In order to keep the error $e(t)$ small an error squared non-negative, the integral of the expression $0.5\, e'(t)Q(t)e(t)$ should be non-negative and small. Thus, the matrix $Q(t)$ must be positive semi-definite. Due to the quadratic nature of the Weightage, large errors require more attention than small errors.
2. The control weighted matrix $R(t)$: The quadratic nature of the control cost expression $0.5u'(t)R(t)u(t)$ indicates that larger control effect requires higher cost. Since the cost of the control has to be a positive quantity, the matrix $R(t)$ should be positive definite.
3. The control signal $u(t)$: The assumption that there are no constraints on the control $u(t)$ is very important in obtaining the closed loop optimal configuration.
4. The terminal cost weighted matrix $F(t_f)$: The main purpose of this term is to ensure that the error $e(t)$ at the final time t_f is small as possible. To guarantee this, the corresponding matrix $F(t_f)$ should be positive semi-definite.
   Further, without loss of generality, we assume that the weighted matrices $Q(t)$, $R(t)$ and $F(t)$ are symmetric. The quadratic performance index has attractive features:
   - It provides an elegant procedure for the design of closed-loop optimal controller.
   - It results in the optimal feedback control that is linear in state function.
5. Infinite Final time : When the final time t_f is infinity, the terminal cost term involving $F(t_f)$ must be zero.

Closed-loop Optimal Control

To formulate a closed-loop optimal control i.e. to obtain the optimal control $u^*(t)$

$$u^*(t) = -R^{-1}(t)\, B'(t)\, P(t)\, x^*(t)$$

which is now a negative feedback of the state $x^*(t)$. This negative feedback resulted from ‘theoretical development’ or ‘mathematics’ of optimal control procedure and not introduced intentionally .Fig. 1 shows a closed loop optimal control system with infinite final time which means that the system needs to be follow the desired signal for the whole processing time.

Matrix Differential Riccati Equation

P(t) is not dependent on the initial state which satisfies the matrix differential equation

$$\dot{P}(t) + P(t)\, A(t) + A'(t)\, P(t) + Q(t) - P(t)\, B(t)\, R^{-1}(t)\, B'(t)\, P(t) = 0$$

This is the matrix differential equation of the Riccati type and often called the matrix differential Riccati Equation (DRE). Also, P(t) is called the Riccati coefficient matrix or simply Riccati matrix or Riccati coefficient and the optimal control (feedback) law can be given as

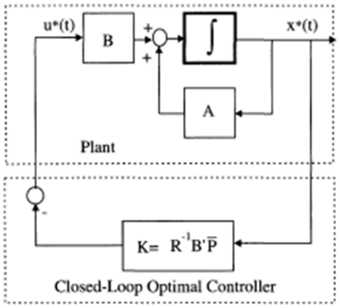

Figure 1: Implementation of closed-loop optimal control: Infinite final time

$$u^*(t) = - R^{-1}(t)\, B'(t)\, P(t)\, x^*(t)$$

The matrix DRE can also be written in a compact form as

$$\dot{P}(t) = - P(t)\, A(t) - A'(t)\, P(t) - Q(t) + P(t)\, E(t)\, P(t)$$

Where $E(t) = B(t)\, R^{-1}(t)\, B'(t)$

Symmetric Property of the Riccati Coefficient Matrix: An important property of Riccati matrix P(t) is that n x n matrix P(t) is symmetric for all $[t_0,t_f]$ i.e. P(t)= P'(t). Ref. [8] The matrices $F(t_f)$, Q(t) and R(t) are symmetric and therefore, the matrix $B(t)\, R^{-1}(t)\, B'(t)$ is also symmetric.

Salient features of state regulator system and matrix differential Riccati equation

1. Riccati Coefficient P(t) : P(t) is a time-varying matrix which depends upon the system matrices A(t) and B(t), the performance index (design) matrices Q(t), R(t) and $F(t_f)$ and the terminal time t_f but P(t) does not depend upon the initial state $x(t_0)$ of the system.
2. P(t) is symmetric and hence it follows that the n x n order matrix DRE represents a system of n(n+1)/2 first order non-linear, time-varying, ordinary differential equations.
3. Optimal Control: The optimal control u*(t) is minimum (maximum) if the control weighted matrix R(t0 is positive definite (negative definite).
4. Optimal state: Using the optimal control ,
   $u^*(t) = - R^{-1}(t)\, B'(t)\, P(t)\, x^*(t)$
   in state equation $\dot{x}(t) = A\, x(t) + B\, u(t)$
   we have,
   $\dot{x}(t) = [A(t) - B(t)\, R^{-1}(t)\, B'(t)\, P(t)]\, x^*(t) = G(t)\, x^*(t)$
   where
   $G(t) = A(t) - B(t)\, R^{-1}(t)\, B'(t)\, P(t)$
   The solution of this state differential equation along with the initial condition $x(t_0)$ give sthe optimal state x*(t). There is no condition on the closed-loop matrix G(t) regarding stability as long as the finite final time (t_f) system is considered.
5. Optimal Cost: The minimum cost J* is given by
   $J^* = (1/2)\, x^{*\prime}(t)\, P(t)\, x^*(t)$ for $t= [t_0,t_f]$
   Where, P(t) is the solution of matrix DRE and x*(t) is the solution of the closed-loop optimal system.
6. Definiteness of the Matrix P(t) : Since $F(t_f)$ is positive semi-definite, and $P(t_f) = F(t_f)$ this reflects that $P(t_f)$ is positive semi-definite, symmetric matrix.
7. Independence of Riccati Coefficient Matrix : The matrix P(t) is independent of the optimal state x*(t), so that once the system and the cost are specified, i.e. once the system/plant matrices A(t) and B(t) are given, and the performance index matrices $F(t_f)$, Q(t) and R(t), the matrix P(t) can be independently computed before optimal system operates in the forward direction from its initial condition.
8. Implementation of the Optimal Control: Fig.2 represents the block diagram implementing the closed-loop optimal controller as shown below.
   The figure shows clearly that the closed loop optimal controller gets its values of P(t) externally, after solving the matrix DRE backward in time from $t= t_f$ to $t = t_0$ and hence there is no way that teh closed loop optimal control configuration can be implemented.
9. Linear Optimal Control: The optimal feedback control u*(t) given as

$$u^*(t) = - K(t)\, x^*(t) \text{ where, the Kalman gain } K(t) = R^{-1}(t)\, B'(t)\, P(t)$$

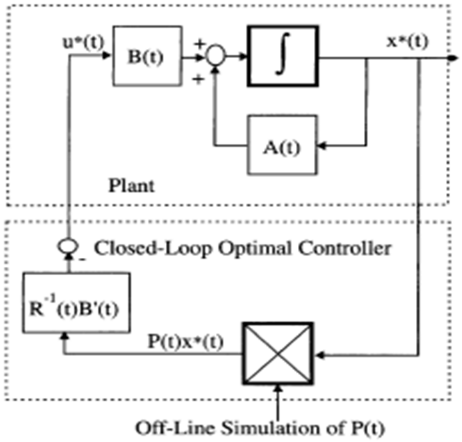

Figure 2: Closed-Loop Optimal Control Implementation

10. Controllability: As long as it is a finite time system, the need of controllability condition on the system for implementing the optimal feedback control is not there because the contribution of those uncontrollable states to the cost function is still a finite quantity only.

State Space Model of DC Motor

A physical plant can be described by a set of linear or non-linear differential equations. Fig.3 depicts the equivalent circuit of a DC motor armature is based on the fact that the armature winding has a resistance R_a, a self inductance L_a, and an induced emf V_b. Ref. [9] In case of a motor , the input energy is electrical energy and the output is the mechanical energy, with an air gap torque T_m at rotational speed ω_m. Armature controlled dc motor uses armature Voltage V_a as the control variable.

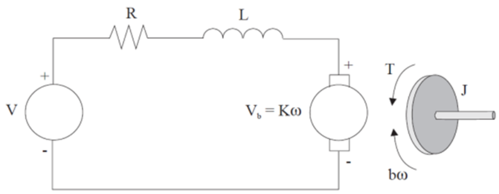

Figure 3: Model of a DC motor

The voltage equation of DC Motor is given as:

$$V_a(s) = R_a\, I_a(s) + L_a\, s\, I_a(s) + V_b\,(s)$$

In steady state , the armature current is constant and the rate of change of armature current is zero. The voltage equation of DC Motor is given as:

$$V_a(s) = R_a\, I_a(s) + V_b\,(s)$$

The air gap power is expressed in terms of the electromagnetic torque and speed as

$$P_a = \omega_m\, T_m = V_b\, I_a$$

Motor torque is given as

$$T_m(s) = K_m\, I_a(s)$$

In case of permanent magnet dc motor Φ is constant because the field winding here is replaced by permanent magnet. K_m is function of permeability of magnetic material and also known as the torque constant.

From Faraday's law,the induced emf if the armature conductors are divided into 'A' parallel paths (neglecting the sign) is

$$V_b = Z\, \Phi_f P\, N_r\, /\, (60\ A)$$

There two possible arrangements of conductors in the armature, wave winding s and lap windings. A= 2 for wave windings and A= P for lap winding (P represents the number of poles of the motor). In compact form we have

$$V_b = K\, \Phi_f\, \omega_m$$

Where $\omega_m = 2\pi N_r/60$ rad/sec and $K = (P/A)Z(1/2\pi)$

If field flux is constant, then the induced emf is proportional to the rotor speed and the constant of proportionality is known as the induced emf constant or back emf constant. Then the induced emf is represented as

$$V_b = K_b \, \omega_m$$

Where K_b is the induced emf constant , given by $K_b = K \, \Phi_f$ Volt/(rad/sec)

For simplicity, the load is modelled as a moment of inertia,J,in Kg-m^2/sec^2, with a viscous friction coefficient B in N-m/(rad/sec). Then the acceleration torque Ta in N-m drives the load and is given by

$$J(d\,\omega_m/dt) + B\,\omega_m = T_m - T_l = T_a$$

Where T_l is the load torque.
The dynamic equations are cast in state space form and are given by
The state equation,

$$\begin{bmatrix} pI_a \\ p\omega_m \end{bmatrix} = \begin{bmatrix} \frac{-R_a}{L_a} & \frac{-K_m}{L_a} \\ \frac{K_m}{J} & \frac{-B}{J} \end{bmatrix} \begin{bmatrix} I_a \\ \omega_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \end{bmatrix} V$$

Where p is the differential operator with respect to time. The above equation is expressed compactly in the form given by $\dot{x} = A\,x + B\,u$ where $x = [I_a \ \omega_m]'$, u =[V], x is the state variable vector and u is the input vector[9].
The output equation is given by,
$\omega_m = [1 \ \ 0] \, \omega_m$

Results

The LQR design minimizes a weighted squared state error and control effort to achieve a state feedback controller design. The optimal feedback state regulation, minimizes the quadratic cost function (performance index)
$J = \int (x^T(t) \, Q \, x(t) + u^T(t) \, R \, u(t))$ for [limits tending from 0 to infinity]
The specifications of separately excited DC Motor are shown in Table 1. The transfer function of the motor in closed loop is obtained from MATLAB script file given as

$$\theta(S) / Va(S) = \frac{0.023}{0.005 s^2 + 0.01002 s + 0.02356}$$

To control the speed of DC Motor, a traditional PID Controller is designed using trial and error method first. After several iterations, the values of Ki, Kp and Kd are chosen as Kp=10,Ki=10,Kd=5. Speed response of traditional PID controller is shown here in section B. The characteristic equation with PID controller is
0.005 s^3 + 0.01002 s^2 + 0.000559 s = 0
In state space representation the system matrices obtained from the MATLAB Script file and the motor specifications which are as follows
A= [-0.003 2.3 ; -0.046 -2]
B= [0;2]
C= [1 0]
D= [0]
Weighting matrices Q and R are selected as Q = [0.1 0,0 0.0001]' and R[0.1]. The step response of closed loop optimal control DC motor is shown in section C. Table 2 lists the performance of PID controller and LQR along with open loop system MATLAB Script file is used to describe the system performance. In order to compare the results of closed loop optimal controller with closed loop system and PID controller with MATLAB environment is used.
The analysis from section D and table 2 shows that the motor performance with optimal control is best suited because the performance measures such as settling time is the least for optimal control, no overshoot and zero steady state error. The Riccati coefficient, P obtained is, P = [0.0772 0.0487; 0.0487 0.0400], which is positive semi-definite, symmetric matrix. This Riccati coefficient is further used to obtain the optimal state x*(t) (here Ia i.e. Armature current and ω_m i.e. angular speed of the armature). State feedback gain matrix, K= [0.9749 0.8009]. The feedback gain matrix is then used to obtain the optimal control u*(t) i.e. V (Armature voltage). Desired pole location at which system is optimal is, E = [-1.8024 + 1.1630i;-1.8024 - 1.1630i]. The optimal performance index can then be obtained from the optimal state x*(t) and Riccati coefficient P. It is clear from the results that the motor speed tracks the desired signal with good transient response. Here the results of the control system are response for step change in desired signal.

Table 1. Separately excited DC Motor Specifications

DC motor specifications		
Particulars	Value	Unit
Moment Of Inertia, J	0.01	Kg-m^2
Back emf constant & Torque constant, Ka	0.023	Nm/A
Viscous friction constant, b	0.00003	Nms
Armature resistance, R	1	ohm
Armature Inductance, L	0.5	Henry

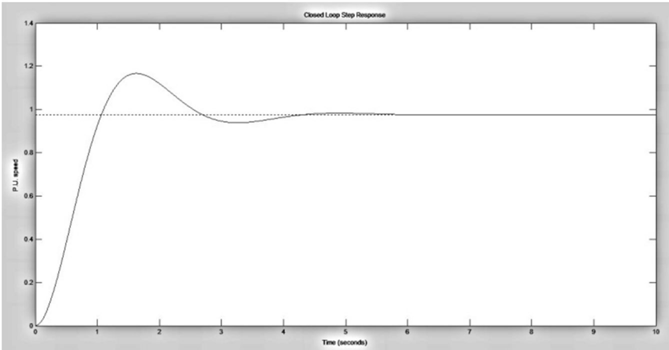

Figure 4. Step response of closed loop system

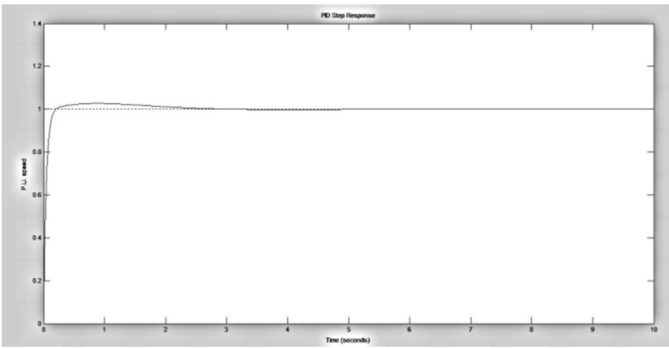

Figure 5. Step response of closed loop system with PID control action

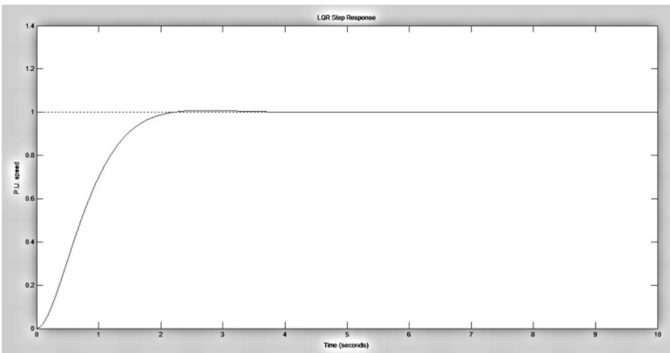

Figure 6. Step response of closed loop optimal control system

Table 2. Comparison of Results

System Performance	Comparison		
	Closed loop system	Closed loop system with PID controller	Closed loop optimal controller
Settling time, Ts	5.77 sec	3.19 sec	2.24 sec
Maximum Overshoot, Mp	Amplitude-1.17 At 1.62 sec	Amplitude – 1.03 At 0.748 sec	0
Steady state error,e_{ss}	0.023	0	0

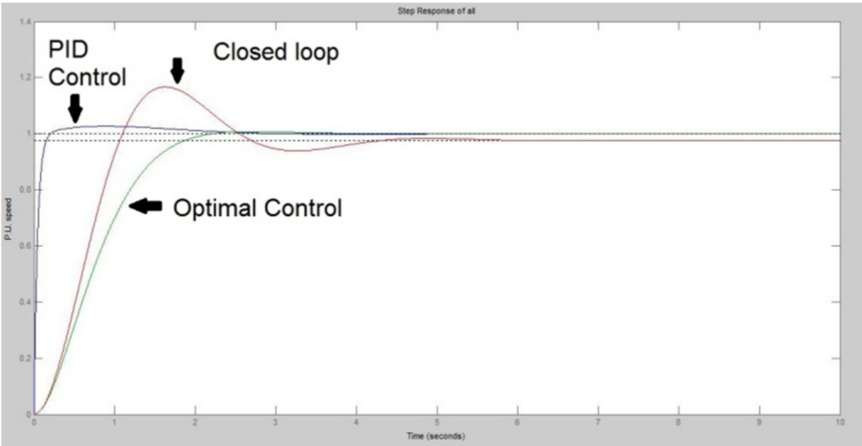

Figure 7. Step response of closed loop, motor with PID control and optimal control

Conclusion

This paper proposes an approach to control design of a DC motor based on LQR control design. The mechanical and electrical parameters of DC motor are used to obtain the response for the system. The LQR design provides an optimal state feedback control that minimizes the quadratic error and control effort. On comparison between the simple closed loop system, closed loop system with PID control action and closed loop optimal controller, the transient response and steady state error in the response due to closed-loop optimal controller yields the best result.

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